## CHAPTER 6: RISK AVERSION AND CAPITAL ALLOCATION TO RISKY ASSETS

## **PROBLEM SETS**

- 1. (e)
- 2. (b) A higher borrowing is a consequence of the risk of the borrowers' default. In perfect markets with no additional cost of default, this increment would equal the value of the borrower's option to default, and the Sharpe measure, with appropriate treatment of the default option, would be the same. However, in reality there are costs to default so that this part of the increment lowers the Sharpe ratio. Also, notice that answer (c) is not correct because doubling the expected return with a fixed risk-free rate will more than double the risk premium and the Sharpe ratio.
- 3. Assuming no change in risk tolerance, that is, an unchanged risk aversion coefficient (A), then higher perceived volatility increases the denominator of the equation for the optimal investment in the risky portfolio (Equation 6.12). The proportion invested in the risky portfolio will therefore decrease.
- 4. a. The expected cash flow is: \$135,000

With a risk premium of 8% over the risk-free rate of 6%, the required rate of return is 14%. Therefore, the present value of the portfolio is \$118,421

b. If the portfolio is purchased for \$118,421, and provides an expected cash inflow of \$135,000, then the expected rate of return [E(r)] is \$135,000

Therefore, E(r) = 14%. The portfolio price is set to equate the expected rate or return with the required rate of return.

- c. If the risk premium over T-bills is now 12%, then the required return is 18%
  The present value of the portfolio is now \$114,407
- d. For a given expected cash flow, portfolios that command greater risk premia must sell at lower prices. The extra discount from expected value is a penalty for risk.
- 5. When we specify utility by  $U = E(r) 0.5A\sigma^2$ , the utility level for T-bills is: 0.07 The utility level for the risky portfolio is: U = 0.12 - 0.0162A"A" must be less than 3.09 for the risky portfolio to be preferred to bills.

(1)	(2)	(3)	(4)	r <sub>Portfolio</sub>	$\sigma_{Portfolio}$	_2
$W_{\mathrm{Bills}}$	$r_{\mathrm{Bills}}$	$W_{\text{Index}}$	r <sub>Index</sub>	$(1) \times (2) + (3) \times (4)$	$(3) \times 20\%$	O Portfolio
0.0	5%	1.0	13.5%	13.5% = 0.135	20% = 0.20	0.0400
0.2	5%	0.8	13.5%	11.8% = 0.118	16% = 0.16	0.0256
0.4	5%	0.6	13.5%	10.1% = 0.101	12% = 0.12	0.0144
0.6	5%	0.4	13.5%	8.4% = 0.084	8% = 0.08	0.0064
0.8	5%	0.2	13.5%	6.7% = 0.067	4% = 0.04	0.0016
1.0	5%	0.0	13.5%	5.0% = 0.050	0% = 0.00	0.0000

10. The portfolio expected return and variance are computed as follows:

11. Computing utility from  $U = E(r) - 0.5 \times A\sigma^2 = E(r) - 1.5\sigma^2$ , we arrive at the values in the column labeled U(A = 3) in the following table:

$W_{\text{Bills}}$	$W_{Index}$	r <sub>Portfolio</sub>	$\sigma_{Portfolio}$	$\sigma^2_{Portfolio}$	U(A = 3)	U(A = 5)
0.0	1.0	0.135	0.20	0.0400	0.0750	0.0350
0.2	0.8	0.118	0.16	0.0256	0.0796	0.0540
0.4	0.6	0.101	0.12	0.0144	0.0794	0.0650
0.6	0.4	0.084	0.08	0.0064	0.0744	0.0680
0.8	0.2	0.067	0.04	0.0016	0.0646	0.0630
1.0	0.0	0.050	0.00	0.0000	0.0500	0.0500

The column labeled U(A = 3) implies that investors with A = 3 prefer a portfolio that is invested 80% in the market index and 20% in T-bills to any of the other portfolios in the table.

12. The column labeled U(A = 5) in the table above is computed from:

 $U = E(r) - 0.5A\sigma^2 = E(r) - 2.5\sigma^2$ 

The more risk averse investors prefer the portfolio that is invested 40% in the market index, rather than the 80% market weight preferred by investors with A = 3.

13. Expected return = 15%

Standard deviation = 19.6%

- 14. Investment proportions:30.0% in T-bills
  - $0.7 \times 25\% = 17.5\%$  in Stock A  $0.7 \times 32\% = 22.4\%$  in Stock B  $0.7 \times 43\% = 30.1\%$  in Stock C

15. Your reward-to-volatility ratio: 
$$S = \frac{18-8}{28} = 0.3571$$
  
Client's reward-to-volatility ratio:  $S = \frac{15-8}{19.6} = 0.3571$ 



17. a. 
$$E(r_C) = r_f + y[E(r_P) - r_f] = 8 + y(18 - 8)$$

If the expected return for the portfolio is 16%, then:

$$16 = 8 + 10 \text{ y} \Rightarrow \text{y} = \frac{16 - 8}{10} = 0.8$$

Therefore, in order to have a portfolio with expected rate of return equal to 16%, the client must invest 80% of total funds in the risky portfolio and 20% in T-bills.

Client's investment proportions:
$$20.0\%$$
 in T-bills $0.8 \times 25\% =$  $20.0\%$  in Stock A $0.8 \times 32\% =$  $25.6\%$  in Stock B $0.8 \times 43\% =$  $34.4\%$  in Stock C

c. 
$$\sigma_{C} = 0.8 \times \sigma_{P} = 0.8 \times 28\% = 22.4\%$$

18. a.  $\sigma_c = y \times 28\%$ 

If your client prefers a standard deviation of at most 18%, then:

y = 64.29% invested in the risky portfolio

- b.  $E(r_c) = 14.429\%$
- 19. a. y\*=0.3644

Therefore, the client's optimal proportions are: 36.44% invested in the risky portfolio and 63.56% invested in T-bills.

- b.  $E(r_c) = 11.644\%$  $\sigma_c = 10.203\%$
- 20. a. If the period 1926 2005 is assumed to be representative of future expected performance, then we use the following data to compute the fraction allocated to equity: A = 4,  $E(r_M) r_f = 8.39\%$ ,  $\sigma_M = 20.54\%$  (we use the standard deviation of the risk premium from Table 6.8). Then y<sup>\*</sup> is given by:

y \*= 0.4972

That is, 49.72% of the portfolio should be allocated to equity and 50.28% should be allocated to T-bills.

b. If the period 1986 - 2005 is assumed to be representative of future expected performance, then we use the following data to compute the fraction allocated to equity: A = 4,  $E(r_M) - r_f = 8.60\%$ ,  $\sigma_M = 16.24\%$  and y\* is given by:

 $y^* = 0.8152$ 

Therefore, 81.52% of the complete portfolio should be allocated to equity and 18.48% should be allocated to T-bills.

- c. In part (b), the market risk premium is expected to be higher than in part (a) and market risk is lower. Therefore, the reward-to-volatility *ratio* is expected to be higher in part (b), which explains the greater proportion invested in equity.
- 21. a.  $E(r_c) = 8\% \implies y = 0.5$ 
  - b.  $\sigma_{\rm C} = 7.5\%$
  - c. The first client is more risk averse, allowing a smaller standard deviation.
- 22. Data:  $r_f = 5\%$ ,  $E(r_M) = 13\%$ ,  $\sigma_M = 25\%$ , and  $r_f^B = 9\%$

The CML and indifference curves are as follows:





23. For y to be less than 1.0 (so that the investor is a lender), risk aversion (A) must be large enough such that:

 $\Rightarrow$  A > 1.28

For y to be greater than 1.0 (so that the investor is a borrower), risk aversion must be small enough such that:

 $\Rightarrow A < 0.64$ 

For values of risk aversion within this range, the client will neither borrow nor lend, but instead will hold a complete portfolio comprised only of the optimal risky portfolio:

y = 1 for  $0.64 \le A \le 1.28$ 

- 24. a. The graph for Problem 22 has to be redrawn here, with:  $E(r_P) = 11\%$  and  $\sigma_P = 15\%$ 
  - b. For a lending position: A > 2.67For a borrowing position: A < 0.89Therefore, y = 1 for  $0.89 \le A \le 2.67$



25. The maximum feasible fee, denoted f, depends on the reward-to-variability ratio. For y < 1, the lending rate, 5%, is viewed as the relevant risk-free rate, and we solve for f as follows:

$$\frac{11-5-f}{15} = \frac{13-5}{25} \implies f = 6 - \frac{15 \times 8}{25} = 1.2\%$$

For y > 1, the borrowing rate, 9%, is the relevant risk-free rate. Then we notice that, even without a fee, the active fund is inferior to the passive fund because:

$$\frac{11-9}{15} = 0.13 < \frac{13-9}{25} = 0.16$$

More risk tolerant investors (who are more inclined to borrow) will not be clients of the fund even without a fee. (If you solved for the fee that would make investors who borrow indifferent between the active and passive portfolio, as we did above for lending investors, you would find that f is negative: that is, you would need to *pay* investors to choose your active fund.) These investors desire higher risk-higher return complete portfolios and thus are in the borrowing range of the relevant CAL. In this range, the reward-to-variability ratio of the index (the passive fund) is better than that of the managed fund.

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- 26. a. Slope of the CML =  $\frac{13-8}{25} = 0.20$ The diagram follows.
  - b. My fund allows an investor to achieve a higher mean for any given standard deviation than would a passive strategy, i.e., a higher expected return for any given level of risk.



27. a. With 70% of his money invested in my fund's portfolio, the client's expected return is 15% per year and standard deviation is 19.6% per year. If he shifts that money to the passive portfolio (which has an expected return of 13% and standard deviation of 25%), his overall expected return becomes:

$$E(r_{c}) = r_{f} + 0.7[E(r_{M}) - r_{f}] = 8 + [0.7 \times (13 - 8)] = 11.5\%$$

The standard deviation of the complete portfolio using the passive portfolio would be:

$$\sigma_{\rm C} = 0.7 \times \sigma_{\rm M} = 0.7 \times 25\% = 17.5\%$$

Therefore, the shift entails a decrease in mean from 14% to 11.5% and a decrease in standard deviation from 19.6% to 17.5%. Since both mean return *and* standard deviation decrease, it is not yet clear whether the move is beneficial. The disadvantage of the shift is that, if the client is willing to accept a mean return on his total portfolio of 11.5%, he can achieve it with a lower standard deviation using my fund rather than the passive portfolio.

To achieve a target mean of 11.5%, we first write the mean of the complete portfolio as a function of the proportion invested in my fund (y):

$$E(r_C) = 8 + y(18 - 8) = 8 + 10y$$

Our target is:  $E(r_c) = 11.5\%$ . Therefore, the proportion that must be invested in my fund is determined as follows:

$$11.5 = 8 + 10y \Rightarrow y = \frac{11.5 - 8}{10} = 0.35$$

The standard deviation of this portfolio would be:

$$\sigma_{\rm C} = y \times 28\% = 0.35 \times 28\% = 9.8\%$$

Thus, by using my portfolio, the same 11.5% expected return can be achieved with a standard deviation of only 9.8% as opposed to the standard deviation of 17.5% using the passive portfolio.

b. The fee would reduce the reward-to-volatility ratio, i.e., the slope of the CAL. The client will be indifferent between my fund and the passive portfolio if the slope of the after-fee CAL and the CML are equal. Let f denote the fee:

Slope of CAL with fee 
$$=\frac{18-8-f}{28} = \frac{10-f}{28}$$
  
Slope of CML (which requires no fee) $=\frac{13-8}{25} = 0.20$ 

Setting these slopes equal we have:

$$\frac{10-f}{28} = 0.20 \Rightarrow 10 - f = 28 \times 0.20 = 5.6 \Rightarrow f = 10 - 5.6 = 4.4\% \text{ per year}$$

28. a. The formula for the optimal proportion to invest in the passive portfolio is:

$$y^* = \frac{E(r_M) - r_f}{A\sigma_M^2}$$

Substitute the following:  $E(r_M) = 13\%$ ;  $r_f = 8\%$ ;  $\sigma_M = 25\%$ ; A = 3.5:

$$y^* = \frac{0.13 - 0.08}{3.5 \times 0.25^2} = 0.2286$$

b. The answer here is the same as the answer to Problem 27(b). The fee that you can charge a client is the same regardless of the asset allocation mix of the client's portfolio. You can charge a fee that will equate the reward-to-volatility *ratio* of your portfolio to that of your competition.

## **CFA PROBLEMS**

1. Utility for each investment =  $E(r) - 0.5 \times 4 \times \sigma^2$ 

We choose the investment with the highest utility value.

Investment	Expected return	Standard deviation	Utility
	E(r)	σ	U
1	0.12	0.30	-0.0600
2	0.15	0.50	-0.3500
3	0.21	0.16	0.1588
4	0.24	0.21	0.1518

- 2. When investors are risk neutral, then A = 0; the investment with the highest utility is Investment 4 because it has the highest expected return.
- 3. (b)
- 4. Indifference curve 2
- 5. Point E
- 6.  $(0.6 \times \$50,000) + [0.4 \times (-\$30,000)] \$5,000 = \$13,000$
- 7. (b)
- 8. Expected return for equity fund = T-bill rate + risk premium = 6% + 10% = 16%Expected return of client's overall portfolio =  $(0.6 \times 16\%) + (0.4 \times 6\%) = 12\%$ Standard deviation of client's overall portfolio =  $0.6 \times 14\% = 8.4\%$

9. Reward-to-volatility ratio = 
$$\frac{10}{14} = 0.71$$

b. With insurance coverage for the full value of the house, costing \$200, end-of-year wealth is certain, and equal to:

 $[(\$50,000 - \$200) \times 1.06] + \$200,000 = \$252,788$ 

Since wealth is certain, this is also the certainty equivalent wealth of the fully insured position.

c. With insurance coverage for  $1\frac{1}{2}$  times the value of the house, the premium is \$300, and the insurance pays off \$300,000 in the event of a fire. The investment in the safe asset is \$49,700. By year end, the investment of \$49,700 will grow to: \$49,700 × 1.06 = \$52,682

The probability distribution of end-of-year wealth is:

	Probability	Wealth
No fire	0.999	\$252,682
Fire	0.001	\$352,682

For this distribution, expected utility is computed as follows:

$$E[U(W)] = [0.999 \times \ln(252,682)] + [0.001 \times \ln(352,682)] = 12.4402205$$

The certainty equivalent is:

 $W_{CE} = e^{12.440222} = $252,766.27$ 

Therefore, full insurance dominates both over- and under-insurance. Over-insuring creates a gamble (you actually gain when the house burns down). Risk is minimized when you insure exactly the value of the house.